

Zitterbewegung and gravitational Berry phase

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Abstract

Berry phases mix states of positive and negative energy in the propagation of fermions and bosons in external gravitational and electromagnetic fields and generate Zitterbewegung oscillations. The results are valid in any reference frame and to any order of approximation in the metric deviation.

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1. Introduction

The contribution of external gravitational fields to the solution of covariant wave equations is contained in a Berry phase [1]. This should be expected because in general relativity the space of parameters of Berry's theory coincides with space-time. The wave equations for fermions and bosons [2], [3], [4], [5] have been solved exactly to first order in the metric deviation $\gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ for any metric and the solutions give the correct Einstein deflection when applied to geometrical optics and can be used in interferometry, gyroscopy, in the study of neutrino helicity and flavour oscillations [6] and of spin-gravity coupling [7]. They also reproduce a variety of known effects like those discussed in [8], [9], [10],[11].

It is shown below that the gravitational Berry phase gives rise to a field-dependent Zitterbewegung (ZB) in the propagation of particles in a gravitational background.

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2. Dirac and Klein-Gordon equations

Consider first the covariant Dirac equation

$$[i\gamma^\mu(x)\mathcal{D}_\mu - m]\Psi(x) = 0. \quad (1)$$

The notations are those of [6]. The first order solutions of (1) are of the form

$$\Psi(x) = \hat{T}(x)\psi(x), \quad (2)$$

where $\psi(x)$ is a solution of the flat space-time Dirac equation

$$(i\gamma^{\hat{\mu}}\partial_\mu - m)\psi(x) = 0, \quad (3)$$

here a plane wave of four-momentum k^α satisfying the relation $k_\alpha k^\alpha = m^2$, and $\gamma^{\hat{\mu}}$ are the usual constant Dirac matrices. The operator \hat{T} is given by [6]

$$\hat{T} = -\frac{1}{2m}(-i\gamma^\mu(x)\mathcal{D}_\mu - m)e^{-i\Phi_T}, \quad (4)$$

$$\Phi_T = \Phi_S + \Phi_G, \quad \Phi_S(x) = \int_P^x dz^\lambda \Gamma_\lambda(z), \quad (5)$$

and

$$\begin{aligned} \Phi_G(x) = & -\frac{1}{4} \int_P^x dz^\lambda [\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)] [(x^\alpha - z^\alpha)k^\beta - (x^\beta - z^\beta)k^\alpha] + \\ & \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z)k^\alpha, \end{aligned} \quad (6)$$

where Γ_λ represents the spin connection. The solutions $\psi(x)$ of (3) can include wave packets, if so desired. In this case the ZB decays in time [12], which is not an essential point in what follows. In (5) and (6), the path integrals are taken along the classical world line of the particle starting from a reference point P .

In most applications $\psi(x)$ is represented by a positive energy solution $\psi(x) = u(\vec{k})e^{-ik_\mu x^\mu}$. However the influence of negative energy solutions $\psi^{(1)}(x) = v(\vec{k})e^{ik_\mu x^\mu}$ can not be neglected because the wave functions $\psi(x)$ by themselves do not form a complete set. A relationship between $\Psi(x)$ and $\Psi^{(1)}(x) = \hat{T}_1\psi^{(1)}(x)$ must therefore be found. The spin-up (\uparrow) and spin-down (\downarrow) components of the spinors u and v obey the well-known equations

$$u_\downarrow = \gamma^5 v_\uparrow, \quad v_\downarrow = \gamma^5 u_\uparrow. \quad (7)$$

The required relation between $\Psi(x)$ and $\Psi^{(1)}(x)$ follows from (7), or simply from the replacement of $\psi(x)$ with $\gamma^5\psi(x)$ in (3). If, in fact, $\Psi(x) = e^{-ik_\mu x^\mu} \hat{T} u$ is a solution of (1), it then follows from (7), the relations $\{\gamma^5, \gamma^{\hat{\mu}}\} = 0$, $\sigma^{\hat{\alpha}\hat{\beta}} = \frac{i}{2}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$, $\gamma^\mu(x) = e_\alpha^\mu(x)\gamma^{\hat{\alpha}}$, $\Gamma_\mu(x) = -\frac{1}{4}\sigma^{\hat{\alpha}\hat{\beta}}e_\alpha^\nu e_{\nu\hat{\beta};\mu}$, and $[\gamma^5, \Gamma^\mu] = 0$ that $\Psi^{(1)}(x) = e^{ik_\mu x^\mu} \hat{T}_1 v$ also is a solution of (1) and $\hat{T}_1 = \gamma^5 \hat{T} \gamma^5$. It is useful to further isolate the gravitational contribution in the vierbein components by writing $e_\alpha^\mu \simeq \delta_\alpha^\mu + h_\alpha^\mu$, which leads to

$$\hat{T} = \frac{1}{2m} \left\{ (1 - i\Phi_G) (m + \gamma^{\hat{\alpha}} k_\alpha) - i (m + \gamma^{\hat{\alpha}} k_\alpha) \Phi_S + \left(k_\beta h_\alpha^\beta + \Phi_{G,\alpha} \right) \gamma^{\hat{\alpha}} \right\} \equiv \hat{T}_0 + \hat{T}_G, \quad (8)$$

where $\hat{T}_0 \equiv \frac{1}{2m} (m + \gamma^{\hat{\alpha}} k_\alpha)$ and \hat{T}_G contains the gravitational corrections. The operator \hat{T}_1 can be immediately calculated from (8).

The gravitational field mixes the positive and negative energy solutions of (3). In fact the eigenstates $U^\pm = 1/\sqrt{2}(u \pm v)$ of γ^5 and the eigenstates u and v of \hat{T}_0 are not the same and \hat{T} , \hat{T}_1 mix u and v . The mixing is effected by \hat{T}_G and \hat{T}_{1G} which are entirely due to Berry phase.

Mixing manifests itself as follows.

The state of a fermion in a gravitational field can be written in the form

$$|\Phi(t)\rangle = \alpha(t)|\psi(t)\rangle + \beta(t)|\psi^{(1)}(t)\rangle = \alpha_0 \hat{T}(t)|\psi(t)\rangle + \beta_0 \hat{T}_1(t)|\psi^{(1)}(t)\rangle, \quad (9)$$

where $|\alpha_0|^2 + |\beta_0|^2 = 1$, from which one obtains

$$\begin{aligned} \alpha(t) &= \langle \psi | \Phi(t) \rangle = \alpha_0 \langle \psi | \hat{T} | \psi \rangle + \beta_0 \langle \psi | \hat{T}_1 | \psi^{(1)} \rangle; \\ \beta(t) &= \langle \psi^{(1)} | \Phi(t) \rangle = \alpha_0 \langle \psi^{(1)} | \hat{T} | \psi \rangle + \beta_0 \langle \psi^{(1)} | \hat{T}_1 | \psi^{(1)} \rangle. \end{aligned} \quad (10)$$

If at $t = 0$ the gravitational field is not present, then $\hat{T}_G = 0$, $\hat{T}_{1G} = 0$ and $\alpha(0) \equiv \alpha_0$, $\beta(0) \equiv \beta_0$. It follows from (10) that as the system propagates in a gravitational field, shifts from $|\psi(t)\rangle$ to $|\psi^{(1)}(t)\rangle$ produce oscillations. Thus the geometrical structure of space-time, represented by gravity, affects Hilbert space by producing oscillations between the positive and negative energy states.

The presence of an electromagnetic field [13] can be accommodated by adding the term qA_α , where q is the charge of the particle, to $\Phi_{G,\alpha}$ in \hat{T} and \hat{T}_1 . The relationship between external electromagnetic fields and ZB has been investigated extensively by Feshbach and Villars [14] for both Dirac and Klein-Gordon equations.

In order to obtain the transition probabilities $|\alpha(t)|^2, |\beta(t)|^2$ from (10) in a concrete case, one can choose for simplicity

$$\psi(x) = f_{0,R} e^{-ik_\alpha x^\alpha} = \sqrt{\frac{E+m}{2m}} \left(\begin{array}{c} f_R \\ \frac{\sigma^3 k}{E+m} f_R \end{array} \right) e^{-ik_\alpha x^\alpha}, \quad (11)$$

where f_R is the positive helicity eigenvector. The normalizations are $\langle \psi | \psi \rangle = 1$, where $\langle \psi | = \langle \psi^\dagger | \gamma^0$, $\langle \psi^{(1)} | \psi^{(1)} \rangle = -1$ and $\langle \psi | \psi^{(1)} \rangle = \langle \psi^{(1)} | \psi \rangle = 0$. In addition, one needs explicit expressions of the metric components for the purpose of calculating \hat{T} and \hat{T}_1 . The choice of the metric

$$\gamma_{00} = 2\phi, \quad \gamma_{ij} = 2\phi \delta_{ij}, \quad (12)$$

where $\phi = -\frac{GM}{r}$, and M, R , are mass and radius of the source, is again dictated by simplicity. The vierbein components to order $\mathcal{O}(\gamma_{\mu\nu})$ are given by

$$e_i^0 = 0, \quad e_0^0 = 1 - \phi, \quad e_i^j = 0, \quad e_k^l = (1 + \phi) \delta_k^l. \quad (13)$$

Without loss of generality, one may consider particles starting from $z = -\infty$, and propagating along $x = b \geq R, y = 0$ in the field of the gravitational source and set $k^3 \equiv k$ and $k^0 \equiv E$.

Returning to (10), if originally the system is in a positive energy state, then $\alpha_0 = 1, \beta_0 = 0$, $|\Phi(t)\rangle = \hat{T}|\psi\rangle$ and from (9) and $\Phi_{G,3} = (E^2/k + k)\phi$ one gets

$$\beta(t) = \frac{e^{-2iq_\alpha x^\alpha}}{2m} \left\{ -\langle \psi^{(1)} | \left[E h_0^0 \gamma^0 + \left(-k h_3^3 + \left(\frac{E^2}{k} + k \right) \phi(z) \right) \gamma^3 \right] | \psi \rangle \right\}, \quad (14)$$

where $q_0 \equiv E$ because the field does not depend on time, hence energy is conserved, and $q_i \equiv k_i^{(i)} - k_i^{(f)}$. The first two terms in (14) are due to Γ_μ and refer to Φ_S . The remaining two terms come from $\Phi_{G,3}$ and are also Berry phase contributions. Thus, according to (10) and (14), the propagation of the particle has two overlapping components: one in which the state of the particle does not change, the other in which oscillations take place from and to energy states of opposite sign with a frequency $2E$, or in ordinary units $2E/\hbar$. This is at least as large as the ZB frequency $2m$. The particle therefore behaves as if it were trying to conserve energy-momentum and angular momentum during its propagation. The presence of the gravitational Berry phase translates into a ZB that vanishes when there is no gravity

acting on the particle and is therefore due to a real force, as pointed out in [14] for the case of an external electromagnetic field. Because the approach is covariant, the result holds true in any frame of reference. Moreover, the non-local potential $K_\lambda(x, x_0) = \Phi_{G,\lambda}(x, x_0) + \Gamma_\lambda(x)$ can be calculated to any order, meaning that a ZB also exists at any order.

The transition amplitude $\langle \psi | \hat{T} | \psi \rangle$ can be better calculated using the relation $\langle \psi | \hat{T} | \psi \rangle = \int_{\lambda_0}^\lambda \langle \psi | \dot{x}^\mu \partial_\mu \hat{T} | \psi \rangle d\lambda$, where $\dot{x}^\mu = k^\mu/m$ and λ is an affine parameter along the particle world line. The calculation is outlined in [6].

The probability of the transition $\psi \rightarrow \psi^{(1)}$ follows from (14) and is

$$P_{\psi \rightarrow \psi^{(1)}} = |\beta(t)|^2 = \left[\frac{1}{2m^2} (k^2 - \frac{E^3}{k}) \right]^2 \phi^2(z). \quad (15)$$

If $\alpha_0 = 0, \beta_0 = 1$, then $|\alpha(t)|^2$ represents the probability for the inverse process $\psi^{(1)} \rightarrow \psi$. One finds

$$\begin{aligned} P_{\psi^{(1)} \rightarrow \psi} &= |\alpha(t)|^2 = |\langle \psi | \hat{T} | \psi^{(1)} \rangle|^2 = |\langle \psi | \gamma^5 \hat{T}_1 \gamma^5 | \psi^{(1)} \rangle|^2 = \\ &= |\langle \psi^{(1)} | \hat{T}_1 | \psi \rangle|^2 = P_{\psi \rightarrow \psi^{(1)}}. \end{aligned} \quad (16)$$

According to (15) and (16), the transitions proceed in both directions with the same probability, as expected.

As mentioned above, an external electromagnetic field can be introduced by simply adding the corresponding Berry phase to (14). The additional term in curly brackets is therefore $\langle \psi^{(1)} | -qA_\mu \hat{\gamma}^\mu | \psi \rangle$. If the addition corresponds to an electromagnetic wave of amplitude f and frequency ω , in vanishing gravity, there is a resonance at $\omega = 2E$ that leads to $|\beta(t)|^2 = (qkf/m^2\omega)^2 \cos^2(2Ex_0)$. If gravity is also present, the resonance condition becomes $\omega = 2E$, $C = qkf/m^2\omega \equiv A$, with C represented by the terms of (14) in curly brackets, and $|\beta(t)|^2 = (A/m)^2 \sin^2(Et)$. The prospects of achieving resonance in laboratory conditions in the near future do not appear favourable.

ZB appears to be universal in condensed matter physics and is the subject of recent, intense research [15]. It is in this area that lie the best opportunities to observe ZB.

Entirely similar conclusions can be reached for the covariant Klein-Gordon equation

$$(g^{\mu\nu} \nabla_\mu \nabla_\nu + m^2) \Phi(x) = 0 \equiv \hat{\mathcal{T}} \Phi(x), \quad (17)$$

which has the first order solution

$$\Phi(x) = e^{-i\Phi_G} \varphi(x), \quad (18)$$

where $\varphi(x)$ satisfies the Klein-Gordon equation in flat space-time

$$(\eta^{\mu\nu} \partial_\mu \partial_\nu + m^2) \varphi(x) = 0 \equiv \hat{\mathcal{T}}_0 \varphi(x). \quad (19)$$

Following the procedure of [14] one can write the plane wave solutions of (19) as

$$\varphi^+(x) = e^{-ip_\mu x^\mu} \chi^+(p), \quad \varphi^-(x) = e^{ip_\mu x^\mu} \chi^-(p), \quad (20)$$

where $\chi^\pm(p)$ are known functions of p . Representing a generic state of the system $\Lambda(x)$ in terms of the free-field solutions φ^\pm one sees immediately that $\Lambda(x)$ is not an eigenstate of $\hat{\mathcal{T}}_0$ and that, therefore, the gravitational part $\hat{\mathcal{T}} - \hat{\mathcal{T}}_0$ due to Φ_G mixes the states of positive and negative energy.

Similar results can be obtained for all known relativistic wave equations.

3. Summary and discussion

It was shown in [14] that static electric and magnetic fields in flat space-time excite a field-dependent ZB. This result has been extended, in this work, not only to electromagnetic fields of any type in curved space-time, but also to any gravitational fields of weak to intermediate strength. The extension is based on the notion of Berry phase. Since the approach is covariant, the result holds true in any reference frame. Moreover, the gauge potential $K_\lambda(x, x_0)$ exists to any order, hence the results remain valid to any order of approximation in $\gamma_{\mu\nu}$ for both fermions and bosons.

Particle propagation is affected by gravitational and electromagnetic Berry phases. They imply gauge structures that mix the field-free states giving rise to oscillations of frequency at least as high as $2m$. This action can be interpreted, in the gravitational case, as an example of how the curvature of space-time can affect Hilbert space by determining transitions between states of positive and negative energy. The transitions involve \hbar . Though resonance conditions between ZB and the external fields exist in principle, their realization for particles in vacuum seems unlikely at present. The significance of the results is related to the role played by Berry phase and the related potential $K_\lambda(x, x_0)$ in the mixing of positive and negative energy states that are necessarily contained in the eigenfunctions of relativistic particles. ZB oscillations appear as the particles strive to conserve energy-momentum and angular momentum along their world lines.

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